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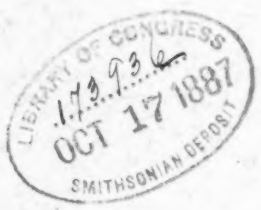

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ANNALS OF MATHEMATICS.

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SOME PROPERTIES OF REPETENDS.

By MR. WILLIAM E. HEAL, Marion, Indiana.

1. In the present paper I investigate some properties of repetends arising from the development of the fraction M/N, where N is not generally a prime number, but M and N are any whole numbers prime to each other. The investigation will not be confined to the decimal system of notation, and when a number is expressed in any notation other than the decimal, it will be enclosed in brackets, with the base expressed in the decimal notation as a subscript. For example, $15 = [17]_8 = [13]_{12}$. When all the numbers in any investigation have been expressed in the adopted notation, the subscripts may be omitted; otherwise when no subscript is expressed, the decimal notation will always be understood. The base will generally be denoted by the letter B.

2. The number of numbers less than and prime to a given number k, has been called by Prof. Sylvester the totient of k, and he proposes for it the notation $\tau(k)$. Each of the numbers whose enumeration makes up the totient may be called a totitive of k. If $k = a^l b^m \ldots h^z$ we know from the Theory of Numbers that

$$\tau(k) = a^{l-1} b^{m-1} \ldots h^{z-1} (a-1)(b-1) \ldots (h-1).$$

3. LEMMA.—If A and k are prime to each other, the difference $A^{\tau(k)} - 1$ is divisible by k; and if m is the least number for which $A^m - 1$ is divisible by k, then m is a divisor of $\tau(k)$.

Let A be any one of the totitives of k, and form the series,

$$1, A, A^2, A^3, \ldots . \qquad (1)$$

Since the terms of this series when divided by k cannot give more than $\tau(k)$ different remainders, we must have, on continuing the series, a term congruous to some one of the preceding terms; suppose that A^m is the first term which possesses this property. We must have

$$A^m \equiv 1, \mod k.$$

For suppose $$A^m \equiv A^n, \mod. k.$$

$$A^m - A^n = A^n (A^{m-n} - 1) \equiv 0, \mod. k;$$

$$\therefore A^{m-n} - 1 \equiv 0, \mod. k.$$

That is A^m is not, as supposed, the first term congruous to a preceding term.

If the remainders of the series,

$$1, A, A^2, A^3, \ldots A^m, \tag{2}$$

include all the totitives of k, we have $m = \tau(k)$.

$$\therefore A^{\tau(k)} - 1 \equiv 0, \mod. k.$$

If $\tau(k) > m$, let B be a totitive of k not included in the series (2), and form the series,

$$B, BA, BA^2, BA^3, \ldots BA^m. \tag{3}$$

No one of the terms of series (3) can be congruous to any one of series (2); for suppose

$$BA^n \equiv A^p, \mod. k;$$

$$\therefore B \equiv A^{p-n}, \mod. k,$$

which is contrary to the hypothesis. Neither can two terms of series (3) be congruous; for suppose

$$BA^n \equiv BA^p, \mod. k;$$

$$\therefore A^n \equiv A^p, \mod. k,$$

which is contrary to the hypothesis, p and n being both less than m. Therefore series (3) includes m totitives of k; therefore $\tau(k) = 2m$ or $\tau(k) > 2m$.

If $\tau(k) > 2m$, let C be a totitive of k not included in series (2) or (3), and form the series,

$$C, CA, CA^2, CA^3, \ldots CA^m. \tag{4}$$

By reasoning upon series (4) as upon series (3), we can show that we have m totitives of k, and therefore $\tau(k) = 3m$, or $\tau(k) > 3m$.

Finally, when all the totitives of k are exhausted,

$$\tau(k) = qm.$$

And since $$A^m \equiv 1, \mod. k,$$

$$A^{qm} = A^{\tau(k)} \equiv 1, \mod. k.$$

4. THEOREM I.—If the base of notation contain all the prime factors of N, the fraction M/N reduces to an exact decimal, and the number of places is equal

to the greatest integer in $(n + n' - 1)/n'$, where $n - n'$ is the greatest difference found by subtracting the exponent of each prime factor in N from the exponent of the same factor in the base.

Let $N = a^t b^u \ldots h^z$, $B = \lambda a^{t'} b^{u'} \ldots h^{z'}$, $i =$ the greatest integer in $(n + n' - 1)/n'$. Then, since $in' = n$, or $in' > n$, and $(i - 1)n' < n$, we have B^i, but not B^{i-1}, divisible by N;

i. e.
$$\frac{MB^i}{N} = q.$$

5. THEOREM II.— If the base contain no prime factor of N the fraction M/N reduces to a repetend.

For by the lemma,
$$B^{\tau(N)} - 1 \equiv 0, \mod N;$$
$$\therefore \frac{MB^{\tau(N)}}{N} = q + \frac{M}{N}.$$

That is, after $\tau(N)$ places the figures repeat.

6. THEOREM III.— If the base contain some but not all the prime factors of N, the fraction M/N reduces to a mixed decimal, and the number of places in the non-repeating part is determined as in Theorem I.

Let
$$N = a^t b^u \ldots f^w h^z, \; B = \lambda a^{t'} \ldots h^{z'}.$$

We have
$$\frac{M}{N} = \frac{MB^{m+i}}{a^t b^u \ldots f^w h^z} = \frac{MM'B^m}{b^u \ldots f^w},$$

where $M' = B^i \div a^t \ldots h^z$, and i has the same meaning as in Theorem I.

In what follows we suppose the base prime to N.

7. THEOREM IV.— The number of places in the period of the repetend M/N is a divisor of $\tau(N)$.

For if
$$\frac{MB^n}{N} = q + \frac{M}{N},$$

we have
$$M(B^n - 1) \equiv 0, \mod N,$$

or since M is prime to N,
$$B^n - 1 \equiv 0, \mod N.$$

Therefore by the lemma, n is a divisor of $\tau(N)$.

EXAMPLES:—
$$\tfrac{1}{63} = .\dot{0}15873\dot{}$$
$$\tau(63) = 3(3-1)(7-1) = 36.$$

The number of places is 6, a divisor of $\tau(63)$.

$$\tfrac{1}{77} = [\tfrac{1}{65}]_{12} = .\dot{0}1\varphi 537\dot{},$$
$$\tau(65) = (7-1)(\pi - 1) = 50,$$

L. of C.

where, as in all that follows, $\varphi = 10$, $\tau = 11$.

The number of places is 6, a divisor of $\tau(65)$.

$$\tfrac{1}{11} = [\tfrac{1}{13}]_8 = .\dot{0}5\dot{6}4272135\dot{}.$$

In this case the number of places $= \tau(13) = 12$.

8. Repetends in which the number of places in the period is equal to $\tau(N)$ are said to be perfect. This definition of a perfect repetend is equivalent to saying that B is a primitive root of N, and viewed in this light the properties of primitive roots proved in Serret's *Cours d'Algèbre Supérieure*, Arts. 305, 317, 318, and 320 may be at once stated as properties of perfect repetends. They are embraced in the four theorems following: —

9. THEOREM V.— The fraction N^{-1} cannot reduce to a perfect repetend unless $N = p^r$, $N = 2p^r$, or $N = 2^2$, where p is an odd prime number.

10. THEOREM VI.— If p is an odd prime number, and p^{-r} reduce to a perfect repetend, p^{-1} will also reduce to a perfect repetend.

11. THEOREM VII.— If p^{-1}, where p is an odd prime number, reduce to a perfect repetend, p^{-r} will or will not reduce to a perfect repetend, according as $B^{p-1} - 1$ is not, or is, divisible by p^2.

EXAMPLE: — $\frac{1}{7} = .\dot{1}42857\dot{}$ is a perfect repetend and $10^{7-1} - 1$ is not divisible by $7^2 = 49$; therefore 7^{-n} will reduce to a perfect repetend. If $n = 2$, $7^{-2} = \frac{1}{49}$ $= .\dot{0}20408163265306122448979591836734693877551\dot{}$, which is perfect, since $\tau(49)$ $= 7 \times 6 = 42$.

12. THEOREM VIII.— If p^{-r}, where p is an odd prime number, reduce to a perfect repetend, so likewise does $\frac{1}{2}p^{-r}$.

13. If the base B is an odd prime number we may determine *a priori* whether N^{-1} reduces to a perfect repetend. By the preceding theorems it is only necessary to examine whether p^{-1}, where p is an odd prime number, reduces to a perfect repetend. We will consider p a prime number, and have

$$B^{(p-1)} - 1 \equiv 0, \quad \text{mod. } p;$$

$$\therefore \quad B^{\frac{1}{2}(p-1)} - 1 \equiv 0, \quad \text{mod. } p,$$

or

$$B^{\frac{1}{2}(p-1)} + 1 \equiv 0, \quad \text{mod. } p.$$

If

$$B^{\frac{1}{2}(p-1)} - 1 \equiv 0, \quad \text{mod. } p,$$

the repetend p^{-1} is evidently not perfect.

14. It is usual to denote the positive or negative unit, which is the remainder of $B^{\frac{1}{2}(p-1)}$, mod. p, by the symbol (B/p) which is known as "Legendre's symbol." We then have by the Law of Reciprocity

$$\left(\frac{B}{p}\right) = (-1)^{\frac{1}{2}(B-1)\frac{1}{2}(p-1)}\left(\frac{p}{B}\right).$$

And $(B/p) = +1$ is a sufficient condition that p^{-1} should not reduce to a perfect repetend.

EXAMPLE: — Let $B = 7 = [10]_7$, $p = 29 = [41]_7$.

$$\left(\frac{10}{41}\right) = (-1)^{\frac{1}{2}(10-1)\ \frac{1}{2}(41-1)} \left(\frac{41}{10}\right) = \left(\frac{41}{10}\right).$$

It is evident, without expansion, that $41^{\frac{1}{2}(10-1)} \equiv 1$, mod. 10; therefore $\left(\frac{10}{41}\right) = +1$. Therefore the repetend is not perfect. By actual expansion $\frac{1}{41} = .\dot{0}14553\dot{6}$.

If $$B^{\frac{1}{2}(p-1)} + 1 \equiv 0, \quad \text{mod. } p,$$

the repetend will be perfect if B does not satisfy a congruence,

$$B^m - 1 \equiv 0, \quad \text{mod. } p,$$

where m is a divisor of $(p-1)$ and less than $\frac{1}{2}(p-1)$; in other words, if p^{-1} does not reduce to a repetend of m places.

EXAMPLE: — Let $B = 7 = [10]_7$, $p = 23 = [32]_7$.

$$\left(\frac{10}{32}\right) = (-1)^{\frac{1}{2}(10-1)\ \frac{1}{2}(32-1)} \left(\frac{32}{10}\right) = -\left(\frac{32}{10}\right).$$

But evidently $32^{\frac{1}{2}(10-1)} \equiv 1$, mod. 10,

$$\therefore \left(\frac{10}{32}\right) = -1.$$

The only factor of $(32-1)$ less than $\frac{1}{2}(32-1)$ is 2; and since B does not satisfy the congruence $B^2 - 1 \equiv 0$, mod. 32, the repetend $\frac{1}{32}$ is perfect. By expansion, $\frac{1}{32} = .\dot{0}20625113436460415532\dot{3}$.

15. The theory may be extended to the case of B, any odd number, by Jacobi's extension of the Law of Reciprocity. If $B = b_1 b_2 \ldots b_n$, where $b_1, b_2, \ldots b_n$ are the prime factors of B, Jacobi defines the symbol (p/B) by the equation,

$$\left(\frac{p}{B}\right) = \left(\frac{p}{b_1}\right)\left(\frac{p}{b_2}\right)\cdots\left(\frac{p}{b_n}\right).$$

The following equation may then be established: —

$$\left(\frac{B}{p}\right) = (-1)^{\frac{1}{2}(B-1)\ \frac{1}{2}(p-1)} \left(\frac{p}{B}\right).$$

The conditions for determining the character of p^{-1} are the same as those already given.

EXAMPLE: — Let $b_1 = 3$, $b_2 = 5$, $B = 15 = [10]_{15}$, $p = 7 = [7]_{15}$.

$$\left(\frac{10}{7}\right) = (-1)^{\frac{1}{2}(10-1)\ \frac{1}{2}(7-1)} \left(\frac{7}{10}\right) = -\left(\frac{7}{3}\right)\left(\frac{7}{5}\right).$$

But $\left(\frac{7}{3}\right) = +1$, $\left(\frac{7}{5}\right) = -1$, and therefore $\left(\frac{10}{7}\right) = +1$. Therefore the repetend $\frac{1}{7}$ is not perfect. In fact $\frac{1}{7} = .\dot{2}$.

16. The following are all the perfect repetends, $B = 10$, whose denominators are less than 100 : —

$\frac{1}{7} = .\dot{1}42857\dot{}$,

$\frac{1}{17} = .\dot{0}588235294117647\dot{}$,

$\frac{1}{19} = .\dot{0}52631578947368421\dot{}$,

$\frac{1}{23} = .\dot{0}434782608695652173913\dot{}$,

$\frac{1}{29} = .\dot{0}344827586206896551724137931\dot{}$,

$\frac{1}{47} = .\dot{0}212765957446808510638297872340425531914893617\dot{}$,

$\frac{1}{49} = .\dot{0}204081632653061224489795918367346938775\dot{1}$,

$\frac{1}{59} = .\dot{0}169491525423728813559322033898305084745762711864406779661\dot{}$,

$\frac{1}{61} = .\dot{0}163934426229508196721311475409836065573770491803278688524\dot{59}$,

$\frac{1}{97} = .\dot{0}103092783505154639175257731958762886597938144329896907216494845360824742268041237113402061855\dot{67}$.

$\frac{1}{98}$ is rejected although of the form $\frac{1}{2}p^{-r}$ because 2 is a divisor of 10, and therefore $\frac{1}{98}$ is not a pure repetend.

17. The following are all the perfect repetends, $B = 12 = [10]_{12}$, whose denominators are less than $50 = [42]_{12}$: —

$\frac{1}{5} = [\frac{1}{5}]_{12} = .\dot{2}49\dot{7}$,

$\frac{1}{7} = [\frac{1}{7}]_{12} = .\dot{1}86\varphi 35\dot{}$,

$\frac{1}{17} = [\frac{1}{15}]_{12} = .\dot{0}8579214\pi 36429\varphi\dot{7}$,

$\frac{1}{25} = [\frac{1}{21}]_{12} = .\dot{0}5915343\varphi 0\pi 62\varphi 68781\dot{\pi}$,

$\frac{1}{31} = [\frac{1}{27}]_{12} = .\dot{0}478\varphi\varphi 093598166\pi 74311\pi 28623\varphi 5\dot{5}$,

$\frac{1}{41} = [\frac{1}{35}]_{12} = .\dot{0}36190\varphi 6532773979\varphi 9\pi 4\pi 85\varphi 2\pi 2568944824120\dot{7}$,

$\frac{1}{43} = [\frac{1}{37}]_{12} = .\dot{0}342295\varphi 3\varphi\varphi 730\varphi 068456\pi 879926181148\pi 1\pi 5376\dot{5}$,

$\frac{1}{49} = [\frac{1}{41}]_{12} = .\dot{0}2\pi 322547\varphi 05\varphi 644\varphi 9380\pi 908996741\pi 61577128\dot{3}\dot{\pi}$.

18. THEOREM IX.—A number expressed by as many digits $(B - 1)$ as there are places in the repetend N^{-1} is divisible by N.

For by Theorem IV, $B^n - 1$ is divisible by N.

EXAMPLES : — 999999 is divisible by 63 and also by 77 ; $\pi\pi\pi\pi\pi\pi$ is divisible by $[53]_{12}$ and also by $[65]_{12}$; 7777777777 is divisible by $[13]_{12}$.

19. Two numbers M, M' are said to belong to the same group, if $MB^m = M'$,

mod. N. If the repetend N^{-1} is perfect, all the totitives of N belong to the same group; but if the repetend N^{-1} reduce to a repetend of n places, where n is a divisor of $\tau(N)$, the totitives of N divide themselves into $\tau(N)/n$ groups of n numbers each. Evidently,

20. THEOREM X.— The same repetend, by starting at different places, will express the value of all fractions having the same denominator, if the numerators belong to the same group.

21. THEOREM XI.— If in the process of reducing to a repetend, we arrive at a remainder $N - M$, we have half the repetend, and the remaining part may be found by subtracting each figure of the first part from $(B - 1)$.

For if $$\frac{MB^k}{N} = q + \frac{N - M}{N} = (q + 1) - \frac{M}{N},$$

$$\frac{MB^{2k}}{N} = B^k(q + 1) + \frac{MB^k}{N} = (B^k - 1)(q + 1) + \frac{M}{N}. \quad Q.\ E.\ D.$$

Repetends which enjoy the property described in the preceding theorem are called complementary.

22. THEOREM XII.— Every repetend whose period consists of $2k$ places is complementary if the denominator of its generating fraction is prime.

For by Theorem IV,

$$B^{2k} - 1 \equiv 0, \mod. N;$$

i. e. $$(B^k - 1)(B^k + 1) \equiv 0, \mod. N;$$

$$\therefore \quad B^k + 1 \equiv 0, \mod. N.$$

For if $$B^k - 1 \equiv 0, \mod. N,$$

the period of N consists of k places instead of $2k$.

COROLLARY. All perfect repetends are complementary.

23. THEOREM XIII.— If a fraction reduce to a complementary repetend of $4k$ places, its denominator is the sum of two squares.

For in the proof of Theorem XI,

$$\frac{MB^{4k}}{N} = (B^{2k} - 1)(q + 1) + \frac{M}{N};$$

$$\therefore \quad M(B^{4k} - 1) = (B^{2k} - 1)(q + 1)N,$$

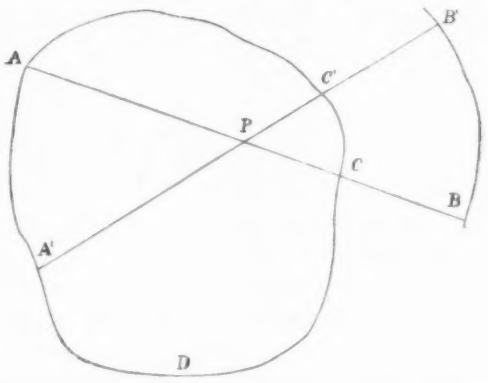
$$\frac{M(B^{2k} + 1)}{N} = q + 1.$$

Therefore N is the sum of two squares, since it is a divisor of the sum of two squares.

ON A THEOREM RELATING TO CLOSED PLANE CURVES.

By Mr. A. D. Risteen, Washington, D. C.

In Williamson's *Calculus* a proof of the following theorem is given: A chord of fixed length l slides round a closed plane curve, its extremities being constantly on the curve. A point on this chord, at a distance a from one end, and b from the other, describes a second closed curve. The area included between these curves is πab.



The following demonstration is interesting on account of a curious fact involved in it, and expressed by equation (2).

Let ACD be the given closed curve, AC the given chord, and B the given point on the chord. Let B' be the position of B when the chord AC has moved to its consecutive position $A'C'$, and let P be the point of intersection of AC and $A'C'$.

Let $AC = l$, $AB = a$, $CB = b$, $PC = x$, and $\angle B'PB = d\theta$. Let u be the area sought, and $BCC'B' = du$. Then

$$du = \tfrac{1}{2}(x + b)^2\, d\theta - \tfrac{1}{2}x^2\, d\theta = bx\,d\theta + \tfrac{1}{2}b^2 d\theta.$$

Integrating between 0 and 2π,

$$u = b\int_0^{2\pi} x\,d\theta + \pi b^2. \qquad (1)$$

We cannot evaluate the integral $\int_0^{2\pi} x\,d\theta$ by ordinary methods, because we do not know the relation between x and θ. We observe, however, that it is independent of b, and that if we can find its value for any particular value of b, the value so found will hold for any other value of b.

If we make $b = 0$, and $\therefore\ u = 0$, (1) leads to the identity $0 = 0$. But putting $b = -l$ and noting that this makes B coincide with A, so that $u = 0$, equation (1) gives us

$$0 = -l\int_0^{2\pi} x\,d\theta + \pi l^2, \quad \text{or} \quad \int_0^{2\pi} x\,d\theta = \pi l. \qquad (2)$$

Substituting this value of the integral in (1), we have

$$u = \pi lb + \pi b^2 = \pi(l + b)\,b = \pi ab.$$

DESCRIPTION OF A CUBICAL INTEGRATOR.

By Assistant Engineer GEO. W. McELROY, U. S. Navy, Columbia, S. C.

qr, *rk*, *kh*, *hq*, *qo*, *ok*, and *hn* are seven rods constituting a modified form of Peaucellier's original parallel motion, the fixed centres being *o* and *n*. The other extremities of the several rods move as shown by dotted lines, the point *r* moving in a straight line, perpendicular to *os*.

FF' and *on* are the two principal pieces to the frame of the instrument. *FF'* carries on its ends two broad wheels, *B* and *B'*, fluted in a manner similar to those on Coradi's planimeter, and thus, as they roll on the paper, the whole instrument moves in a straight line to any desired distance. This result could be accomplished in a number of ways, one being to make *B* and *B'* narrow and compelling them to run in grooves. The device chosen seems, however, to be the best that can be adopted, as it has given remarkably accurate results in the planimeter mentioned.

mm is a disc pivoted on the frame at *o*. *T* is a small wheel on this disc, with finely divided spur teeth on its periphery engaging with corresponding teeth on *B*.

ww is a registering wheel like that on any planimeter.

At the point *o* we have the rods *oq* and *ok* pivoted together with the disc, but each moving independently.

The broken lines (– – – – – –) indicate the mid-position of the system of rods. It is evident that as the instrument rolls on the paper, the circumference of *B*, and hence of *T*, will move through a linear distance equal to that passed over by the instrument. As *ww* rests upon the disc, it is evident that as the instrument rolls along, a certain amount of motion will be communicated by the disc to *ww*. It is also evident that when the point *r* moves in a vertical direction the disc and wheels *B*, *B'* must for that instant remain fixed. Hence, if *r* traces a curve and stops at the starting point, any vertical movement of the point *r* will not affect the reading of the wheel *ww*; for the sum of the decrements from such motion will equal the sum of the increments. The only permanent registration on *ww* will be the algebraic sum of what motion may be communicated to it by the disc *mm* as it revolves back and forth, and that disc can get motion only when *B* rolls on the paper and *r* remains (it may be for an instant only) at a constant distance above *os*.

Let the radius of *T* be *u*.

Also let *y* be the perpendicular distance of *r* above *os*, and let *dx* be the velocity, or rate, of *r* in the direction *os*.

Then T will receive an angular motion,

$$d\theta = \frac{dx}{w}.$$

Draw ρ and ρ_1. Let the angle $qbo = \omega$.

The plane mm passes under ww at a rate $\rho_1 d\theta$, which, for an infinitesimal movement, may be represented by bf perpendicular to ρ_1.

From the figure it is evident that

$$fbw = qbo = \omega = 90^\circ - fbq.$$

Resolving fb along bw, we get

$$fb \cos \omega = \rho_1 d\theta \cos \omega. \tag{1}$$

From the triangle oqb we get

$$c^2 = e^2 + \rho_1^2 - 2e\rho_1 \cos \omega; \tag{2}$$

and from obr

$$\rho^2 = \rho_1^2 + d^2 - 2\rho_1 d \cos (180^\circ - \omega)$$
$$= \rho_1 + d^2 + 2\rho_1 d \cos \omega. \tag{3}$$

From (2) and (3)

$$\cos \omega = \frac{\rho^2 - c^2 - d^2 + e^2}{2\rho_1 (e + d)}. \tag{4}$$

Substituting from (4) in (1), we get as the increment imparted to ww,

$$\frac{\rho^2 - c^2 - d^2 + e^2}{2 (e + d)} d\theta.$$

In integrating between the limits of θ due to one complete oscillation of mm as r describes the curve A, the terms containing the constants will disappear, because θ after reaching its maximum returns to its initial value.

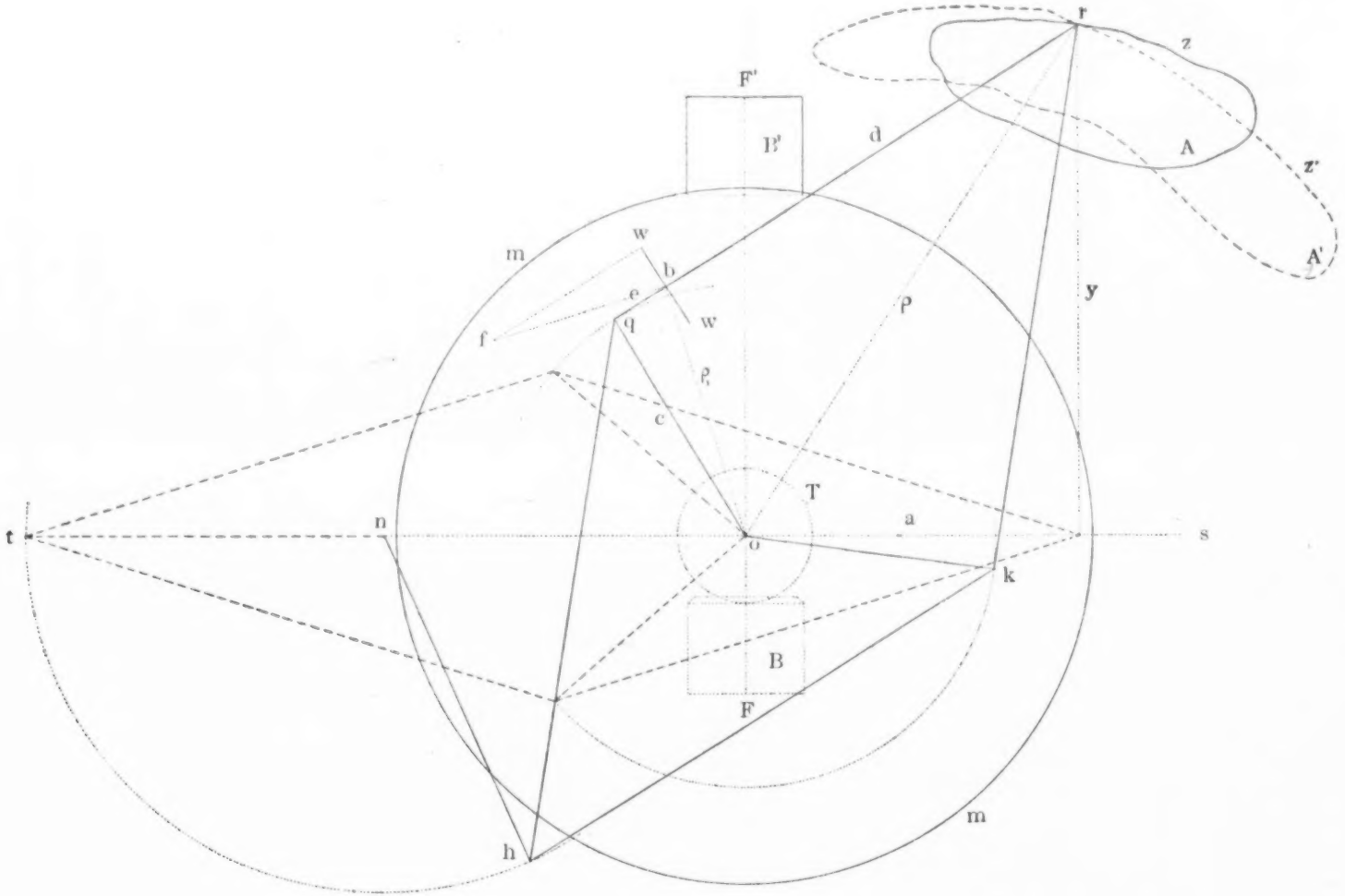
Hence the amount registered in an oscillation will be equal to

$$\frac{1}{2 (e + d)} \int \rho^2 d\theta, \tag{5}$$

taken between proper limits, and will be confined to this value. This is evidently an area, but it is not that of A. It represents the area A' which is enclosed by a curve gotten in this way: any point z' is subject to the conditions that $z'o$ and the angle $z'or$ are, respectively, the values of ρ and the angular motion of mm when r arrives at z.

It will be seen that (5) is independent of the position of ww on qr and of the length oq. In fact, this is the proof for an ordinary planimeter oqr acting on A'. Now $\rho^2 = a^2 + y^2$; whence substituting in (5) we get

$$\frac{1}{2 (e + d)} \int (a^2 + y^2)\, d\theta; \tag{6}$$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but $d\theta = \frac{dx}{u}$; whence (6) becomes

$$\frac{1}{2u(e+d)}\int (a^2 + y^2)\, dx. \qquad (7)$$

When the instrument has completed its forward and backward trip, as much will have been additive in the above integral due to a^2 as is subtractive; hence the definite value of (7) after r has finished the circuit will be due only to

$$\frac{1}{2u(e+d)}\int y^2 dx.$$

Hence ww registers an amount

$$= \frac{\text{volume generated by revolution of } A \text{ about } os}{2u(e+d)\pi},$$

or $$V = 2\pi u(e+d) \times (\text{amount registered on } ww),$$

in which the parenthesis depends on the diameter of ww and the number of turns it makes. If e and u are in inches, the parenthesis will also have to be in inches and V will be in cubic inches.

This proves, then, that when r has gone entirely around the curve enclosing A an amount has been registered on the measuring wheel ww equal to the volume that would be generated by A revolving about os as an axis.

Such an instrument would be useful in getting the cubical contents, and hence the weight, of certain parts of machinery, all machinery in fact that has been turned up in lathes, or cast symmetrically with respect to some axis. In most of these cases the lower part of the curve would coincide with os.

It could also be used, supposing oqr to be made detachable, to find the position of the centre of gravity of A. Thus: find the volume as shown; then detach oqr and find the area; from these two the position of the centre of gravity above os is determined by applying Guldin's theorem. Turn the instrument quarter way round, and we find the distance of the centre of gravity from an axis at right angles to os. These two distances fix its position.

I have shown y as falling a little inside the disc mm. In a practical instrument the disc should be small enough to allow this so that r could, if necessary, move down to, or below, os.

ON THE GENERAL LINEAR DIFFERENTIAL EQUATION.

By Prof. J. E. Oliver, Ithaca, N. Y.

Forsyth* gives the convergent series,

$$v = S_0 - S_1 + S_2 - \ldots,$$

as a general solution of the equation,

$$D^2v + \mu v = 0;$$

wherein $S = C + C'x$, and $S_k = \int_0^x \int_0^x \mu S_{k-1}\, dx^2$.

An obvious generalization would be as follows: —

To solve the equation

$$D^n y + X_1 D^{n-1} y + \ldots + X_{n-1} Dy + X_n y = X, \qquad (1)$$

wherein $X_1, \ldots X_n, X$ are functions of x only (or constants), let us operate upon (1_1)† with $D^{-n} = \int_0^x \int_0^x \ldots \int_0^x (\,)\, dx^n$; and let us operate upon (1_2) with $\int\int \ldots \int (\,)\, dx^n$, which differs from D^{-n} only by admitting n arbitrary constants of integration, $C_1, \ldots C_n$, and may be written

$$D^{-n}(\,) + C_1 \cdot \frac{x^{n-1}}{(n-1)!} + \ldots + C_{n-1} \cdot x + C_n.$$

Or if only a particular solution of (1) be desired, we may give convenient values to $C_1, \ldots C_n$; e. g. making $C_1 = \ldots = C_n = 0$, we may operate on both members of (1) with D^{-n}.

The first-named operators give

$$y + D^{-n} \cdot X_1 D^{n-1} y + \ldots + D^{-n} \cdot X_{n-1} Dy + D^{-n} \cdot X_n y = \int \ldots \int X dx^n, \qquad (2)$$

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† By (1_1) and (1_2), respectively, I mean the first and second member of (1); by (3_{14}), the equation $(3_1) = (3_4)$; etc.

$$\therefore \quad y = [1 + \{D^{-n}.X_1 D^{n-1}() + \ldots + D^{-n}.X_n()\}]^{-1}.\int\ldots\int X dx^n$$

$$= \int\ldots\int X dx^n - \{\ldots\}\int\ldots\int X dx^n + \{\ldots\}^2\int\ldots\int X dx^n - \ldots$$

$$= [1 - \{\ldots\} + \{\ldots\}^2 - \ldots]$$

$$\times [D^{-n}X + C_1\frac{x^{n-1}}{(n-1)!} + \ldots + C_{n-1}x + C_n]; \qquad (3)$$

wherein $\{\ldots\}[D^{-n}X + \ldots + C_n] = D^{-n}.X_1 D^{n-1}[D^{-n}X + \ldots + C_n]$
$+ \ldots + D^{-n}.X_n[D^{-n}X + \ldots + C_n]$,

and $\{\ldots\}^k[D^{-n}X + \ldots + C_n] \equiv \{D^{-n}.X_1 D^{n-1}() + \ldots + D^{-n}.X_n()\}$
$\times \{\ldots\}^{k-1}[D^{-n}X + \ldots + C_n]$.

That (3_{14}) is a primitive of (1), appears by retracing the steps so as to get (1) from (3_{14}). Thus (1_1) is got from (2_1) and (1_2) from (2_2) by the same unideterminate operator D^n, to which both D^{-n} and $\int\ldots\int()\,dx^n$ are inverse; which, of course, justifies our having at first got (2_1) from (1_1) and (2_2) from (1_2) by different operators. Also, the primitive (3_{14}) is complete, having n effective arbitrary constants $C_1, \ldots C_n$.

To show the convergence of (3_4), let l be the length of the path followed by x in each of the n integrations implied in D^{-n}*; and let L_k, A, C_0 be positive real constants such that all along this path of x all the moduli of $\frac{l^k}{k!}, \frac{l^{k+1}}{(k+1)!}, \ldots$ are $< L_k$, and all those of $X_1, \ldots X_n$ are $< A$, and that of X is $< C_0$. Then, since no definite single integral can be larger† than the product of the length of its variable's path into the largest value taken by the function to be integrated, it will be seen that each term of (3_4), such as

$$\pm S_k, \quad = \pm\{\ldots\}^k[D^{-n}X + C_1\frac{x^{n-1}}{(n-1)!} + \ldots + C_{n-1}x + C_n],$$

is smaller than if all the integrations had been from $x = 0$ to $x = l$, and $X_1, \ldots X_n$ had each been constant and $= A$, and X had been the constant C_0. But in that case S_k would have reduced to

$$A^k(D^{-1} + \ldots + D^{-n})^k(C_0.D^{-n}1 + \ldots + C_{n-1}.D^{-1}1 + C_n),$$

$$< C_0.A^k n^k L_{k+n} + C_1.A^k n^k L_{k+n-1} + \ldots + C_{n-1}.A^k n^k L_{k+1} + C_n.A^k n^k L_k.$$

*Unless it happen that the path must be lengthened to avoid regions where not all of $X_1, \ldots X_n$ X are holomorphic, l may be taken = mod. x.

† If $\mathcal{N} = A + Bi = Re^{\theta i}$ and $\mathcal{N}' = A' + B'i = R'e^{\theta' i}$, wherein $i = \sqrt{-1}$ while $A, A', B, B', R, R', \theta, \theta'$ are all real, and R, R' are not negative; then let us say that $\mathcal{N}$ is $\left\{\begin{matrix} greater \\ less \end{matrix}\right.$ than $\mathcal{N}'$, or $\mathcal{N} \gtrless \mathcal{N}'$, when $A \gtrless A'$; but that $\mathcal{N}$ is $\left\{\begin{matrix} larger \\ smaller \end{matrix}\right.$ than $\mathcal{N}'$ when $R \gtrless R'$. E. g., $-3 \pm i$ is larger but less than $2 \pm 2i$.

Hence (3_4) is not less convergent than

$$C_0 \cdot \sum_{k=1}^{k=\infty} A^k n^k L_{k+n} + \ldots + C_n \cdot \sum_{k=1}^{k=\infty} A^k n^k L_k,$$

wherein the series multiplied by any constant C_r converges onward from $k = Aln - n + r$, slowly at first but finally very fast.

In the same way, if $X_1 = \ldots = X_{m-1} = 0$, (3_4) is, at worst, convergent from $k = lm^{-1}[A(n - m + 1)]^{1/m} - (n + 1)m^{-1}$ onward, and the convergence will usually proceed about m times as fast as if $X_1 \ldots$ were not 0. To secure this advantage, we can, with Forsyth,* first transform (1) by the substitution $y = ze^{-\frac{1}{n}\int X_1 dx}$ into $D^n z + P_2 D^{n-2} z + \ldots + P_n z = P$; but the transformation may be unadvisable if $\ldots X_n, X$ be already in good form for repeated integration.

In estimating the final rate of convergence, A can safely be replaced by the largest of all the values assumed by mod. X_1, mod. $X_2^{\frac{1}{2}}$, $\ldots$ mod. $X_n^{1/n}$ during the integrations: e. g. if the series (3_4) is to be used only for values of x smaller than a value R, we need consider only the largest of the values of mod. $X_1, \ldots$ mod. $X_n^{1/n}$ when mod. $x <$ mod. R.

Forms of $X_1, \ldots X_n, X$ for which the successive integrations are rapid and produce terms that readily combine, will of course occur to the reader; but the only cases of much interest will be those where the law of the series (3_4) is clear.

The method may fail in the rare case where no path lies from 0 to x except across a line where not all of $X_1, \ldots X_n, X$ are finite.

* Ibid.

ON THE DIFFERENTIAL EQUATION

$$\frac{dy}{dx} + y^2 + Py + Q = 0.$$

By Prof. W. W. Johnson, Annapolis, Md.

1. The peculiarity of the equation here considered, in which P and Q are functions of x, is its reducibility to a linear form of the second order, namely,

$$\frac{d^2w}{dx^2} + P\frac{dw}{dx} + Q = 0,$$

by the transformation $y = \frac{1}{w}\frac{dw}{dx}$. The equation of the first order includes Riccati's equation, which may, without loss of generality, be written

$$\frac{dy}{dx} + y^2 = cx^m;$$

and the reduction to the linear form has been extensively used in the treatment of that equation, for example, in Cayley's solution.* The present paper presents the relation between the equations in their general forms from quite a different point of view.

2. Starting with the linear equation of the second order

$$\frac{d^2w}{dx^2} + P\frac{dw}{dx} + Q = 0, \tag{1}$$

and denoting $\frac{d}{dx}$ by D, let the equation be written in the symbolic form

$$(D^2 + P.D + Q)w = 0; \tag{2}$$

and let the attempt be made to solve the equation, by resolving the operator into symbolic factors of the first order: for if we can reduce the equation to the form

$$[D - \varphi(x)][D - f(x)]w = 0, \tag{3}$$

we can, by the successive solution of two linear equations of the first order, express w as a function of x. Denote the function $f(x)$ by y. It is readily seen that $-\varphi(x)$ must then be $P + y$, so that the equation is

$$(D + P + y)(D - y)w = 0, \tag{4}$$

in which y denotes an unknown function of x. This equation is to be identical with equation (2); expanding, it becomes

$$D^2w + P.Dw - (Dy + Py + y^2)w = 0,$$

* *Phil. Mag.*, Fourth Series, Vol. 36, pp. 348–351.

$$\frac{dy}{dx} + y^2 + Py + Q = 0.$$

and comparing this with equation (2) we have for the determination of y

$$\frac{dy}{dx} + y^2 + Py + Q = 0, \tag{5}$$

an equation of the first order, but not linear.

3. Suppose now that y is any particular integral of equation (5); then if w satisfies the equation

$$(D - y)\, w = 0, \tag{6}$$

it will plainly satisfy equation (4), that is to say, equation (1). Solving equation (6), we find

$$w = ce^{\int y dx}. \tag{7}$$

Thus to every particular integral of equation (5) there corresponds an integral of equation (1), and if we have two particular integrals of (5), y_1 and y_2, we have two *independent* integrals of (1), and therefore its complete integral

$$w = c_1 e^{\int y_1 dx} + c_2 e^{\int y_2 dx},$$

without the necessity of solving successively two linear equations as indicated in equation (4). But, if we possess only one particular integral y_1 of (5), then w_1 is defined by (7), and by putting in (4)

$$(D - y_1)\, w = v, \tag{8}$$

we have $$(D + P + y_1)\, v = 0;$$

whence $$v = c_2 e^{-\int P dx} e^{-\int y_1 dx} = \frac{c_2}{w_1} e^{-\int P dx};$$

and finally, by the solution of the linear equation (8), we obtain the complete integral

$$w = c_1 w_1 + c_2 w_1 \int e^{-\int P dx} \frac{dx}{w_1^2},$$

in which the second integral is expressed in its usual form in terms of the integral w_1, supposed known.

4. Equation (6) may be written

$$y = \frac{1}{w}\frac{dw}{dx} = \frac{d}{dx}\log w, \tag{6}$$

thus expressing the integral of equation (5) in terms of that of the linear equation (1). Because, therefore, the integral of the latter equation is of the form

$$w = c_1 Y_1 + c_2 Y_2,$$

that of the former is of the form

$$y = \frac{c_1 X_1' + c_2 X_2'}{c_1 X_1 + c_2 X_2},$$

in which X_1 and X_2 are functions of x, and X_1', X_2' their derivatives; or, putting c in place of the ratio $c_2 : c_1$,

$$y = \frac{X_1' + cX_2'}{X_1 + cX_2}, \tag{10}$$

which contains, as it should, but one arbitrary constant. The mode in which the constant c enters the complete integral is the same as in the special case of Riccati's equation, and is the characteristic of the form of differential equation under consideration.

5. The equation (5) is no easier of solution in any given case than the linear equation (1) of the second order; so that in general, as well as in the special case of Riccati's equation, it is best treated in the linear form. Some of the properties of the linear equation are, however, more readily derived from the non-linear form (5), which might be called the *cognate equation* of the first order. For example, if from the equation

$$\frac{dy}{dx} + y^2 + Py + Q = 0, \tag{5}$$

we wish to remove the term Py, by a change of dependent variable, we obviously must assume

$$y = v - \tfrac{1}{2}P; \tag{11}$$

and making the substitution we have

$$\frac{dv}{dx} - \frac{1}{2}\frac{dP}{dx} + v^2 - Pv + \tfrac{1}{4}P^2 + Pv - \tfrac{1}{2}P^2 + Q = 0,$$

or

$$\frac{dv}{dx} + v^2 + Q_0 = 0, \tag{12}$$

where

$$Q_0 = Q - \tfrac{1}{4}P^2 - \tfrac{1}{2}\frac{dP}{dx}. \tag{13}$$

Consider now the equation to which (12) is cognate; viz:—

$$\frac{d^2u}{dx^2} + Q_0 u = 0, \tag{14}$$

the relation between u and v being, as in (6),

$$v = \frac{d}{dx}\log u.$$

Equation (11) gives

$$\log v = \log u - \tfrac{1}{2}\int P dx,$$

so that $$z = e^{-\frac{1}{2}\int P dx} u,$$

is the transformation which converts

$$\frac{d^2z}{dx^2} + P\frac{dz}{dx} + Q = 0 \qquad (1)$$

into the form (14) which lacks the second term, and Q_0 has the value given in equation (13).

SOLUTIONS OF EXERCISES.

60 and 61

DIVIDE a sphere by a secant plane into two parts whose surfaces (volumes) shall be as 1 : 3.

SOLUTION.

Let the secant plane divide the diameter normal to it in the ratio $m : n$. The heights and surfaces of the two zones are respectively

$$\frac{n}{m+n}D, \quad \frac{m}{m+n}D; \quad \text{and} \quad \frac{\pi n}{m+n}D^2, \quad \frac{\pi m}{m+n}D^2.$$

The area of the common base of the two segments is

$$\frac{\pi mn}{(m+n)^2}D^2.$$

Which values under the given condition, give

1. For the surfaces,

$$3\left[\frac{\pi nD^2}{m+n} + \frac{\pi mnD^2}{(m+n)^2}\right] = \frac{\pi mD^2}{m+n} + \frac{\pi mnD^2}{(m+n)^2};$$

from which $$\left(\frac{m}{n}\right)^2 - 4\left(\frac{m}{n}\right) - 3 = 0,$$

$$\frac{m}{n} = \sqrt{7} + 2 = 4.64575+.$$

2 For the volumes, the mid-areas being

$$\frac{\pi n D^2}{4}\frac{2m+n}{(m+n)^2} \quad \text{and} \quad \frac{\pi m D^2}{4}\frac{2n+m}{(m+n)^2},$$

the mean areas are

$$\frac{\pi n D^2}{6(m+n)^2}(3m+n) \quad \text{and} \quad \frac{\pi m D^2}{6(m+n)^2}(3n+m);$$

whence
$$3\frac{\pi n^2 D^2(3m+n)}{6(m+n)^2} = \frac{\pi m^2 D^2(3n+m)}{6(m+n)^2},$$

from which
$$\left(\frac{m}{n}\right)^3 + 3\left(\frac{m}{n}\right)^2 - 9\frac{m}{n} - 3 = 0,$$

$$\frac{m}{n} = 2.06416+.$$

The condition $n < m < (m+n)$ selects the roots. [*W. H. Echols.*]

109

EACH of four equal circles touches two of the others. Find the average area included between them. [*Artemas Martin.*]

SOLUTION.

Let A, B, C, D be the centres of the four equal circles, r their radius, and let $\varphi = \angle BAC = \angle BDC$; then $AB = AC = BD = CD = 2r$, and area of triangle $BAC = 2r^2 \sin\varphi$ = area of triangle BDC. Since the sum of the angles $A, B, C, D = 4\pi$, the sum of the areas of the four sectors $= \pi r^2$. Hence the area enclosed by the circles is $4r^2 \sin\varphi - \pi r^2$. The limits of φ are $\frac{1}{2}\pi$ and $\frac{1}{3}\pi$. Let J = the average area required, and we have

$$J = \int_{\frac{1}{3}\pi}^{\frac{1}{2}\pi}(4r^2\sin\varphi - \pi r^2)\,d\varphi \div \int_{\frac{1}{3}\pi}^{\frac{1}{2}\pi} d\varphi$$

$$= \frac{6r^2}{\pi}\int_{\frac{1}{3}\pi}^{\frac{1}{2}\pi}(4\sin\varphi - \pi)\,d\varphi$$

$$= \frac{6r^2}{\pi}\Big[-4\cos\varphi - \pi\varphi\Big]_{\frac{1}{3}\pi}^{\frac{1}{2}\pi}$$

$$= \frac{r^2}{\pi}(12 - \pi^2).$$

[*Artemas Martin; C. A. Waldo; Henry Heaton; E. L. Stabler.*]

111

A CIRCLE is inscribed in an isosceles triangle; another in the space between the first circle and the vertex, and so on *ad infinitum*. What is the vertical angle when the sum of the areas of the circles is $1/n$ of the area of the triangle?

[*L. G. Weld.*]

SOLUTION.

Let $2a$ be the base, $2x$ the vertical angle, Δ the area of the triangle; ρ the radius, and C the area of the first inscribed circle.

Then
$$\Delta = a^2 \cot x,$$
$$\rho = a \tan (45^\circ - \tfrac{1}{2}x) = a \frac{1 - \sin x}{\cos x},$$
$$C = \pi a^2 \tan^2 (45^\circ - \tfrac{1}{2}x) = \pi a^2 \frac{1 - \sin x}{1 + \sin x}.$$

The second circle is inscribed in a triangle having the same vertical angle $2x$, and a base $2a' = 2a \dfrac{a \cot x - 2\rho}{a \cot x} = 2a \dfrac{1 - \sin x}{1 + \sin x}$. Its area C' is therefore

$$C' = \pi a'^2 \frac{1 - \sin x}{1 + \sin x} = \pi a^2 \left(\frac{1 - \sin x}{1 + \sin x}\right)^3 = C \left(\frac{1 - \sin x}{1 + \sin x}\right)^2.$$

The same reasoning applies to the following circles.

The condition of the problem,
$$\frac{1}{n} \Delta = C + C' + C'' + \ldots,$$
thus becomes
$$\frac{1}{n} a^2 \cot x = \pi a^2 \frac{1 - \sin x}{1 + \sin x} \left[1 + \left(\frac{1 - \sin x}{1 + \sin x}\right)^2 + \left(\frac{1 - \sin x}{1 + \sin x}\right)^4 + \ldots\right]$$
$$= \pi a^2 \frac{\cos^2 x}{4 \sin x};$$

hence
$$\cos x = \frac{4}{n\pi}, \quad \text{or } 2x = 2 \cos^{-1} \frac{4}{n\pi}.$$

[*Alexander Ziwet; L. G. Weld; E. L. Stabler.*]

112

GENERALIZATION II.

To inscribe in a given triangle ABC a triangle PQR whose sides QR, RP, PQ pass respectively through three given points O_p, O_q, O_r.

SOLUTION.

To solve this more general problem, draw through O_p any three straight lines cutting AB in P_1, P_2, P_3 and BC in Q_1, Q_2, Q_3; draw O_qQ_1, O_qQ_2, O_qQ_3, which will meet CA in three points R_1, R_2, R_3; finally, join O_r to R_1, R_2, R_3, and let these joining lines meet AB in P_1', P_2', P_3'. The pencils O_p and O_q, as well as O_q and O_r, being in perspective position, the ranges P, Q, R, P' are projective. The two collinear projective ranges P and P', on AB, will, in general, have two self-corresponding points, which are the solutions of the problem. The construction of the self-corresponding points of two such ranges, is a well-known fundamental problem of Modern Geometry and can always be effected by means of an auxiliary circle.*

Now, let O_p, O_q, O_r be the points at infinity in the directions perpendicular to AB, BC, CA respectively, and the general problem reduces to the particular case proposed by Mr. Lane. In this case, the ranges P, Q, R, P' are *similar*, and, therefore, need only two points each for their determination. We may then proceed as follows: Selecting as the arbitrary points P_1, P_2 on AB, the point B, and the foot of the perpendicular from C on AB, we have at once for Q_1, Q_2 the points B and C, since they are the intersections with BC of the perpendiculars to AB erected in P_1, P_2. The perpendicular to BC in B intersects CA in R_1, while C is the point R_2. Lastly, the perpendiculars to CA erected in R_1 and C meet AB in two points P_1', P_2', which form a range similar to, and collinear with, B, P_2. The self-corresponding point (or centre of similitude) can, of course, be constructed linearly, without the aid of a circle, for instance, as follows: Draw a parallel to AB, and from P_1, P_2 drop perpendiculars on it, meeting it in P_1'', P_2''; join $P_1'P_1''$, $P_2'P_2''$, and from the intersection of these lines drop a perpendicular on AB; it will meet this line in the self-corresponding point of the ranges.

[*Alexander Ziwet.*]

116

FIND the value of $\int Q dx$ where

$$Q = \cos(a_1x + b_1)\cos(a_2x + b_2) \ . \ . \ . \ \cos(a_nx + b_n),$$

$a_1, a_2, \ . \ . \ . \ a_n$, and $b_1, b_2, \ . \ . \ . \ b_n$ being constants. [*A. Hall.*]

SOLUTION.

Applying successively the formulæ,

$$\cos A \cos B = \tfrac{1}{2}[\cos(A + B) + \cos(A - B)],$$

$$\cos(A \pm B)\cos C = \tfrac{1}{2}[\cos(A \pm B + C) + \cos(A \pm B - C)],$$

etc., etc.,

*See, for instance, L. Cremona's *Projective Geometry*, translated by Leudesdorf, Oxford 1885, p. 169.

we obtain

$$Q = (\tfrac{1}{2})^n \Sigma \cos [(a_1 \pm a_2 \ . \ . \ . \pm a_n)\, x + b_1 \pm b_2 \ . \ . \ . \pm b_n],$$

where the signs $\pm$ are independent of one another and include all possible variations; provided, each b has the same sign as the corresponding a; whence immediately,

$$\int Q dx = \frac{1}{2^n (a_1 \pm a_2 \ . \ . \ . \pm a_n)} \Sigma \sin [(a_1 \pm a_2 \ . \ . \ . \pm a_n)\, x + b_1 \pm b_2 \ . \ . \ . \pm b_n].$$

In case, however, any coefficient of x in Q vanishes, we have for the corresponding term in $\int Q dx$

$$(\tfrac{1}{2})^n \int \cos (b_1 \pm b_2 \ . \ . \ . \pm b_n)\, dx = (\tfrac{1}{2})^n x \cos (b_1 \pm b_2 \ . \ . \ . \pm b_n).$$

[*Ormond Stone.*]

122

If $P_1P_2P_3$ be a triangle inscribed in an ellipse, the co-ordinates of the point of concourse of its altitudes are given by the relations,

$$2ax = (a^2 + b^2)(\cos p_1 + \cos p_2 + \cos p_3) - (a^2 - b^2) \cos (p_1 + p_2 + p_3),$$

$$2by = (a^2 + b^2)(\sin p_1 + \sin p_2 + \sin p_3) - (a^2 - b^2) \sin (p_1 + p_2 + p_3);$$

where p_1, p_2, p_3 are the eccentric anomalies of the vertices. Prove these relations, and find the values of x, y when for a given P the area of $P_1P_2P_3$ is greatest. Show that the locus of the orthocentre of all such triangles is also an ellipse.

[*W. M. Thornton.*]

SOLUTION.

1. The abscissa of the circumcentre is, by Exercise 98,

$$x_1 = \frac{c^2}{a} \cos \tfrac{1}{2}(p_2 + p_3) \cos \tfrac{1}{2}(p_3 + p_1) \cos \tfrac{1}{2}(p_1 + p_2)$$

$$= \frac{c^2}{4a} [\cos p_1 + \cos p_2 + \cos p_3 + \cos (p_1 + p_2 + p_3)],$$

by easy reductions. The abscissa of the centroid is

$$x_2 = \tfrac{1}{3} a (\cos p_1 + \cos p_2 + \cos p_3).$$

The orthocentre divides the line joining these points in the ratio 3 : 2 externally; that is,

$$x = 3x_2 - 2x_1$$

$$= a (\cos p_1 + \cos p_2 + \cos p_3) - \frac{c^2}{2a} [\cos p_1 + \cos p_2 + \cos p_3 + \cos (p_1 + p_2 + p_3)],$$

or $2ax = (a^2 + b^2)(\cos p_1 + \cos p_2 + \cos p_3) - (a^2 - b^2) \cos (p_1 + p_2 + p_3)$.

The ordinate can be found in the same way.

2. The area of the triangle (Ex. 98) is

$$\Delta = 2ab \sin \tfrac{1}{2}(p_2 - p_3) \sin \tfrac{1}{2}(p_3 - p_1) \sin \tfrac{1}{2}(p_1 - p_2),$$

the differentiation of which, since p_1, p_2, p_3 are independent of one another, gives, after reduction,

$$\tan \tfrac{1}{2}(p_1 - p_2) = \tan \tfrac{1}{2}(p_2 - p_3) = \tan \tfrac{1}{2}(p_3 - p_1).$$

Whence for the maximum

$$p_1 - p_2 = p_2 - p_3 = p_3 - p_1 = \tfrac{2}{3}n\pi,$$

where n is an arbitrary integer. Consequently

$$\cos p_1 + \cos p_2 + \cos p_3 = \cos p_1 + \cos (p_1 + \tfrac{2}{3}n\pi) + \cos (p_1 + \tfrac{4}{3}n\pi) = 0,$$

$$\sin p_1 + \sin p_2 + \sin p_3 = \sin p_1 + \sin (p_1 + \tfrac{2}{3}n\pi) + \sin (p_1 + \tfrac{4}{3}n\pi) = 0,$$

$$p_1 + p_2 + p_3 = 3p_1 + 2n\pi;$$

and the co-ordinates of the orthocentre are given by the relations,

$$2ax = -(a^2 - b^2) \cos 3p_1,$$

$$2by = -(a^2 - b^2) \sin 3p_1.$$

3. Whence $a^2x^2 + b^2y^2 = \tfrac{1}{4}(a^2 - b^2)^2$, the equation to an ellipse.

[*W. M. Thornton.*]

123

FIND an expression for the area of a quadrilateral inscribed in an ellipse in terms of the eccentric anomalies of its vertices and the axes of the curve.

[*R. H. Graves.*]

SOLUTION 1.

Let the equation to the ellipse be $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$; let the co-ordinates of the vertices taken in order be $(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)$ and let the eccentric angles of the same points be $\alpha, \beta, \gamma, \delta$.

The area of the quadrilateral

$$= \pm \tfrac{1}{2}(y_1x_2 - y_2x_1 + y_2x_3 - y_3x_2 + y_3x_4 - y_4x_3 + y_4x_1 - y_1x_4)$$

$$= \pm \tfrac{1}{2} \begin{vmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ y_1 & y_2 & y_3 & y_4 \\ x_1 & x_2 & x_3 & x_4 \end{vmatrix}$$

$$= \pm \tfrac{1}{2} ab \begin{vmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ \sin\alpha & \sin\beta & \sin\gamma & \sin\delta \\ \cos\alpha & \cos\beta & \cos\gamma & \cos\delta \end{vmatrix}$$

$$= \pm \tfrac{1}{2} ab \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \sin\alpha - \sin\gamma & \sin\beta - \sin\delta & \sin\gamma & \sin\delta \\ \cos\alpha - \cos\gamma & \cos\beta - \cos\delta & \cos\gamma & \cos\delta \end{vmatrix}$$

$$= \pm \tfrac{1}{2} ab \begin{vmatrix} 2\cos\tfrac{1}{2}(\alpha+\gamma)\sin\tfrac{1}{2}(\alpha-\gamma) & 2\cos\tfrac{1}{2}(\beta+\delta)\sin\tfrac{1}{2}(\beta-\delta) \\ 2\sin\tfrac{1}{2}(\alpha+\gamma)\sin\tfrac{1}{2}(\gamma-\alpha) & 2\sin\tfrac{1}{2}(\beta+\delta)\sin\tfrac{1}{2}(\delta-\beta) \end{vmatrix}$$

$$= \pm 2ab \sin\tfrac{1}{2}(\alpha-\gamma)\sin\tfrac{1}{2}(\beta-\delta) \begin{vmatrix} \cos\tfrac{1}{2}(\alpha+\gamma) & \cos\tfrac{1}{2}(\beta+\delta) \\ \sin\tfrac{1}{2}(\alpha+\gamma) & \sin\tfrac{1}{2}(\beta+\delta) \end{vmatrix}$$

$$= \pm 2ab \sin\tfrac{1}{2}(\alpha-\gamma)\sin\tfrac{1}{2}(\beta-\delta)\sin\tfrac{1}{2}(\alpha+\gamma-\beta-\delta).$$

Cor. 1. If $\gamma = \delta$, the expression becomes $\pm 2ab \sin\frac{1}{2}(\alpha-\beta)\sin\frac{1}{2}(\beta-\gamma)\sin\frac{1}{2}(\gamma-\alpha)$, which agrees with the formula for the area of an inscribed triangle given in Salmon's *Conic Sections*, Art. 231, Ex. 5.

Cor. 2. If the vertices lie also on the circumference of a circle, $\alpha+\beta+\gamma+\delta = 2n\pi$.* Then the area is $\pm 2ab \sin\frac{1}{2}(\alpha-\gamma)\sin\frac{1}{2}(\beta-\delta)\sin(\beta+\delta)$.

Cor. 3. If the normals at the four vertices meet in a point, $\alpha+\beta+\gamma+\delta = (2n+1)\pi$.† Then the area is $\pm 2ab \sin\frac{1}{2}(\alpha-\gamma)\sin\frac{1}{2}(\beta-\delta)\cos(\beta+\delta)$.

[*R. H. Graves.*]

SOLUTION II.

From Exercise 98

$$P_1P_3P_2 = \quad 2ab \sin\tfrac{1}{2}(p_1-p_3)\sin\tfrac{1}{2}(p_3-p_2)\sin\tfrac{1}{2}(p_2-p_1),$$

$$P_1P_3P_4 = -2ab \sin\tfrac{1}{2}(p_1-p_3)\sin\tfrac{1}{2}(p_3-p_4)\sin\tfrac{1}{2}(p_4-p_1).$$

But

$$2\sin\tfrac{1}{2}(p_3-p_2)\sin\tfrac{1}{2}(p_2-p_1) = \cos\tfrac{1}{2}(p_1-2p_2+p_3) + \cos\tfrac{1}{2}(p_3-p_1),$$

$$2\sin\tfrac{1}{2}(p_3-p_4)\sin\tfrac{1}{2}(p_4-p_1) = \cos\tfrac{1}{2}(p_1-2p_4+p_3) + \cos\tfrac{1}{2}(p_3-p_1),$$

and

$$\cos\tfrac{1}{2}(p_1-2p_2+p_3) - \cos\tfrac{1}{2}(p_1-2p_4+p_3) = 2\sin\tfrac{1}{2}(p_1-p_2+p_3-p_4)\sin\tfrac{1}{2}(p_2-p_4).$$

Therefore

$$P_1P_2P_3P_4 = 2ab \sin\tfrac{1}{2}(p_1-p_3)\sin\tfrac{1}{2}(p_2-p_4)\sin\tfrac{1}{2}(p_1-p_2+p_3-p_4).$$

*See Smith's *Conic Sections*, Art. 186, Ex. 1.

†Ibid., Art. 198.

The negative sign is given to the value of $P_1P_3P_4$ because the perimeter is described in the sense opposite to that pursued in describing $P_1P_3P_2$.

[*W. M. Thornton.*]

SOLUTION III.

Adopting the notation of Exercise 98 the area of the quadrilateral can be found by adding the triangle $P_1P_2P_3$ to $P_1P_3P_4$; or by the sum of the triangles $P_1P_2P_4$ and $P_2P_3P_4$. Hence we have the two values of the area:

$$A = 2ab \sin \tfrac{1}{2}(p_3 - p_1)\,[\sin \tfrac{1}{2}(p_4 - p_1) \sin \tfrac{1}{2}(p_4 - p_3) + \sin \tfrac{1}{2}(p_2 - p_1) \sin \tfrac{1}{2}(p_3 - p_2)],$$

$$A = 2ab \sin \tfrac{1}{2}(p_4 - p_2)\,[\sin \tfrac{1}{2}(p_4 - p_3) \sin \tfrac{1}{2}(p_3 - p_2) + \sin \tfrac{1}{2}(p_2 - p_1) \sin \tfrac{1}{2}(p_4 - p_1)].$$

We may write therefore

$$A = 2ab \sin \tfrac{1}{2}(p_3 - p_1) \sin \tfrac{1}{2}(p_4 - p_2) \times \frac{\sin \tfrac{1}{2}(p_4 - p_1) \sin \tfrac{1}{2}(p_4 - p_3) + \sin \tfrac{1}{2}(p_2 - p_1) \sin \tfrac{1}{2}(p_3 - p_2)}{\sin \tfrac{1}{2}(p_4 - p_2)}.$$

Now

$$\sin \tfrac{1}{2}(p_4 - p_1) = \sin \tfrac{1}{2}(p_4 - p_2 + p_2 - p_1) = \sin \tfrac{1}{2}(p_4 - p_2) \cos \tfrac{1}{2}(p_2 - p_1) + \cos \tfrac{1}{2}(p_4 - p_2) \sin \tfrac{1}{2}(p_2 - p_1),$$

$$\sin \tfrac{1}{2}(p_4 - p_3) = \sin \tfrac{1}{2}(p_4 - p_2 + p_2 - p_3) = \sin \tfrac{1}{2}(p_4 - p_2) \cos \tfrac{1}{2}(p_2 - p_3) + \cos \tfrac{1}{2}(p_4 - p_2) \sin \tfrac{1}{2}(p_2 - p_3);$$

whence $\sin \tfrac{1}{2}(p_4 - p_1) \sin \tfrac{1}{2}(p_4 - p_3)$

$$= \sin \tfrac{1}{2}(p_4 - p_2)^2 \cos \tfrac{1}{2}(p_2 - p_3) \cos \tfrac{1}{2}(p_2 - p_1)$$
$$+ \sin \tfrac{1}{2}(p_4 - p_2) \cos \tfrac{1}{2}(p_4 - p_2) \cos \tfrac{1}{2}(p_2 - p_3) \sin \tfrac{1}{2}(p_2 - p_1)$$
$$+ \sin \tfrac{1}{2}(p_4 - p_2) \cos \tfrac{1}{2}(p_4 - p_2) \sin \tfrac{1}{2}(p_2 - p_3) \cos \tfrac{1}{2}(p_2 - p_1)$$
$$+ \cos \tfrac{1}{2}(p_4 - p_2)^2 \sin \tfrac{1}{2}(p_2 - p_3) \sin \tfrac{1}{2}(p_2 - p_1).$$

The second and third of these terms give

$$\sin \tfrac{1}{2}(p_4 - p_2) \cos \tfrac{1}{2}(p_4 - p_2) \sin \tfrac{1}{2}(p_2 - p_3 + p_2 - p_1).$$

Changing $\cos^2$ to $1 - \sin^2$ the last term in A is destroyed, and we have from the first and fourth terms of the product,

$$\sin \tfrac{1}{2}(p_4 - p_2)^2 \cos \tfrac{1}{2}(p_2 - p_3 + p_2 - p_1).$$

Dividing out $\sin \tfrac{1}{2}(p_4 - p_2)$, the last factor of A becomes

$$\sin \tfrac{1}{2}(p_4 - p_2) \cos \tfrac{1}{2}(p_2 - p_3 + p_2 - p_1) + \cos \tfrac{1}{2}(p_4 - p_2) \sin \tfrac{1}{2}(p_2 - p_3 + p_2 - p_1),$$

or

$$\sin \tfrac{1}{2}(p_4 - p_3 + p_2 - p_1).$$

Hence the area of the quadrilateral is

$$A = 2ab \sin \tfrac{1}{2}(p_3 - p_1) \sin \tfrac{1}{2}(p_4 - p_2) \sin \tfrac{1}{2}(p_4 - p_3 + p_2 - p_1).$$

[*A. Hall.*]

[Prof. Hall furnished the result contained in Exercise 123 independently of Prof. Graves, but later.]

124

THE area of the curve

$$r^2 = \frac{(a^2 - b^2)^2 \sin^2\theta \cos^2\theta}{a^2 \sin^2\theta + b^2 \cos^2\theta}$$

is one-half that of the circumscribing circle. [*R. H. Graves.*]

SOLUTION I.

It may be shown, from Art. 173 of Williamson's *Differential Calculus*, that the curve is the pedal of the evolute of the ellipse, the centre being the pedal origin; hence its area is $\frac{1}{2}\pi(a - b)^2$.* From the nature of its generation and from the result of Exercise 93 it appears that the radius of its circumscribing circle is $a - b$. Hence the truth of the proposition. [*R. H. Graves.*]

SOLUTION II.

$$2d\Omega = \frac{c^4 \sin^2\theta \cos^2\theta}{a^2 \sin^2\theta + b^2 \cos^2\theta}\, d\theta$$

$$= \frac{c^4 \sin^2\theta \cos^2\theta}{c^2 \sin^2\theta + b^2}\, d\theta$$

$$= c^2 \cos^2\theta\, d\theta + b^2 d\theta \frac{a^2 b^2}{b^2 + c^2 \sin^2\theta}\, d\theta$$

$$= c^2 \cos^2\theta\, d\theta + b^2 d\theta - \frac{a^2 b^2}{a^2 \sin^2\theta + b^2 \cos^2\theta}\, d\theta$$

$$= \tfrac{1}{2}(a^2 + b^2)\, d\theta + \tfrac{1}{2} c^2 \cos 2\theta - \frac{a^2 b^2}{a^2 \tan^2\theta + b^2}\, d\tan\theta;$$

$$\therefore\ 2\Omega = \tfrac{1}{2}(a^2 + b^2)\theta + \tfrac{1}{4}c^2 \sin 2\theta - ab \tan^{-1}\left(\frac{a}{b}\tan\theta\right).$$

The limits for one quadrant are 0 and $\frac{1}{2}\pi$.

Hence
$$\tfrac{1}{2}\Omega = \tfrac{1}{4}\pi(a^2 + b^2) - \tfrac{1}{2}\pi ab$$
$$= \tfrac{1}{4}\pi(a - b)^2;$$
$$\therefore\ \Omega = \tfrac{1}{2}\pi(a - b)^2,$$

which was to be proved. [*Wm. M. Thornton.*]

*See Williamson's *Integral Calculus*, Art. 142.

126

FROM the centre of each of two equal coins a coin is cut at random. If one of these random coins be placed on the other at random in a horizontal position, what is the probability that the top coin will not fall off, supposing one coin just as likely to be placed on top as the other? [*Artemas Martin.*]

SOLUTION.

Let r be the radius of the given equal coins, and x and y the respective radii of the random coins cut from them.

If the coin whose radius is x be placed on top, the chance that it will not fall off is $\frac{\pi x^2}{\pi (x+y)^2} = \frac{x^2}{(x+y)^2}$. If the coin whose radius is y be placed on top, the chance that it will not fall off is $\frac{y^2}{(x+y)^2}$. The total chance, therefore, that the top coin will not fall off for a particular, or given, value of each x and y is $\frac{x^2+y^2}{2(x+y)^2}$.

The probability required is the sum of all the separate probabilities resulting from all possible values of x and y.

$$\therefore \quad p = \int_0^r \int_0^r \frac{(x^2+y^2)\,dxdy}{2(x+y)^2} \div \int_0^r \int_0^r dxdy$$

$$= \frac{1}{2r^2} \int_0^r \int_0^r \frac{(x^2+y^2)\,dxdy}{(x+y)^2}$$

$$= \frac{1}{2r^2} \int_0^r \left[(2x+r) - \frac{2x^2}{x+r} + 2x \log x - 2x \log (x+r) \right] dx$$

$$= \frac{1}{2r^2} \Big[2rx + x^2 \log x - (x^2+r^2) \log (x+r) \Big]_0^r$$

$$= 1 - \log 2. \qquad [\textit{Artemas Martin.}]$$

129

THE two equal sides of an isosceles triangle move on two fixed points. Required the equation of the locus of the centre of gravity of the triangle.

[*Yale Problems.*]

SOLUTION.

The locus of the vertex of the triangle is that segment of the circle, on the

base determined by the two fixed points, which contains the vertical angle of the triangle.

Its centroid lies on the bisectrix of this angle at a constant distance from the vertex.

The bisectrix passes constantly through a fixed point on the circumference of the circle.

Hence the locus is a circular conchoid.

Referred to the fixed point as pole, and the perpendicular bisector of the base of the segment as axis, its polar equation is of the form

$$\rho = 2R\cos\theta - c. \qquad [Wm.\ Symmonds.]$$

132

A BROKEN line, $ABCDE\ldots$ is drawn in a plane, having its angles equal and the concavity always on the same side. Each of the successive parts BC, CD, DE, etc., is half as long as the preceding. The length and direction of AB are given and the common angle. Required the direction and distance from A of the point to which the end of the line approaches, when the construction as described is continued indefinitely. [*Yale Problems.*]

SOLUTION.

Using A as origin, and AB as axis we find the co-ordinates of the terminal point to be

$$x = a + \tfrac{1}{2}a\cos\alpha + \tfrac{1}{4}a\cos 2\alpha + \tfrac{1}{8}a\cos 3\alpha + \ldots,$$

$$y = \tfrac{1}{2}a\sin\alpha + \tfrac{1}{4}a\sin 2\alpha + \tfrac{1}{8}a\sin 3\alpha + \ldots;$$

whence

$$x + iy = a + \tfrac{1}{2}ae^{i\alpha} + \tfrac{1}{4}ae^{2i\alpha} + \tfrac{1}{8}ae^{3i\alpha} + \ldots$$

$$= \frac{a}{1 - \tfrac{1}{2}e^{i\alpha}}$$

$$= \frac{2a}{(2 - \cos\alpha) + i\sin\alpha}$$

$$= 2a\,\frac{(2 - \cos\alpha) + i\sin\alpha}{(2 - \cos\alpha)^2 + \sin^2\alpha}.$$

Therefore

$$x = 2a\,\frac{2 - \cos\alpha}{5 - 2\cos\alpha}, \quad y = 2a\,\frac{\sin\alpha}{5 - 2\cos\alpha},$$

and the required distance and direction are

$$\rho = \sqrt{(x^2 + y^2)} = \frac{2a}{\sqrt{(5 - 2\cos\alpha)}},$$

$$\theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{\sin\alpha}{2 - \cos\alpha}.$$

[*L. G. Weld.*]

134

LET $\rho_a, \rho_b, \rho_c, \rho_d, \rho_e$ be five vectors drawn from a common but undetermined origin to five given points A, B, C, D, E. Find five scalars a, b, c, d, e such that the vector equation

$$a\rho_a + b\rho_b + c\rho_c + d\rho_d + e\rho_e = 0$$

shall hold true for all positions of the origin. [*Yale Problems.*]

SOLUTION.

Projection on the axes gives

$$a(x - x_a) + b(x - x_b) + c(x - x_c) + d(x - x_d) + e(x - x_e) = 0,$$
$$a(y - y_a) + b(y - y_b) + c(y - y_c) + d(y - y_d) + e(y - y_e) = 0,$$
$$a(z - z_a) + b(z - z_b) + c(z - z_c) + d(z - z_d) + e(z - z_e) = 0;$$

relations which must hold for all values of x, y, z.

Therefore
$$ax_a + bx_b + cx_c + dx_d + ex_e = 0,$$
$$ay_a + by_b + cy_c + dy_d + ey_e = 0,$$
$$az_a + bz_b + cz_c + dz_d + ez_e = 0,$$
$$a + b + c + d + e = 0.$$

And a, b, c, d, e must be proportional to the minors of

$$\begin{Vmatrix} x_a & x_b & x_c & x_d & x_e \\ y_a & y_b & y_c & y_d & y_e \\ z_a & z_b & z_c & z_d & z_e \\ 1 & 1 & 1 & 1 & 1 \end{Vmatrix}$$

[*W. M. Thornton.*]

135

A UNIFORM elastic cord weighs 100 ounces. Its length when unstretched is 100 feet. If it is laid on a smooth table and stretched, its length increases one foot for every ounce of force applied. It is thrown over two pullies twenty feet apart, and on the same level, so that the central loop balances the pendent ends. Discuss the curve and the tensions. [*Yale Problems.*]

SOLUTION.

Let H denote the uniform horizontal tension in the arc of the catenary,

w weight of the arc measured from the lowest point to (x, y),

t tension in the arc at (x, y),

s length of the arc whose weight is w.

The equations of equilibrium are

$$\frac{dx}{H} = \frac{dy}{w} = \frac{ds}{t}, \quad t^2 = H^2 + w^2.$$

And since the original length of the arc ds is by hypothesis $ds/(1 + \frac{t}{100})$,

$$ds = dw\,(1 + \tfrac{t}{100}).$$

Hence, we find for the differential equations of the arc

$$dx = \frac{H}{100}\,dw + H\,\frac{dw}{\sqrt{(H^2 + w^2)}},$$

$$dy = \frac{1}{100}\,w dw + \frac{w dw}{\sqrt{(H^2 + w^2)}}.$$

The integral form of these equations is with the origin at the lowest point of the arc

$$x = \frac{Hw}{100} + H \log \frac{w + \sqrt{(H^2 + w^2)}}{H},$$

$$y = \frac{w^2}{200} + \sqrt{(H^2 + w^2)} - H.$$

These define the form of the arc.

To determine H we observe that at the pulley the values of t and w are connected by the relation

$$T + W = 50;$$

$$\therefore \quad W = 25 - \frac{H^2}{100}, \quad T = 25 + \frac{H^2}{100},$$

and the abscissa of the pulley is by hypothesis 10.

Hence we have $$\log H = \log 50 + 0.25 - \frac{10}{H} - \frac{H^2}{10000}.$$

This transcendental equation gives approximately

$$H = 3.407;$$

$$\therefore \quad W = 24.884, \quad T = 25.116.$$

To find the stretched length of the pendent cord, we have from considerations analogous to those used above

$$dP = dT + \tfrac{1}{100}\,T dT,$$

$$P = T + \tfrac{1}{200}\,T^2$$

$$= 28.270.$$

For the length of the half of the stretched arc we have

$$ds = dw + \tfrac{1}{100}\sqrt{(H^2 + w^2)}\, dw,$$

$$S = W + \frac{1}{200} W\sqrt{(H^2 + W^2)} + \frac{H^2}{200} \log \frac{W + \sqrt{(H^2 + W^2)}}{H}$$

$$= W + \frac{1}{200} TW + \frac{1}{200} H^2 \log \frac{50}{H}$$

$$= 28.164.$$

[*W. M. Thornton.*]

EXERCISES.

SELECTED FROM CASEY'S CONIC SECTIONS.

145

FIND the equation to the circle through the feet of the normals from (h, k) to the parabola $y^2 = 2px$.

146

FIND the condition that the chords AB, CD in an ellipse should meet the transverse axis in points equidistant from the centre, the points A, B, C, D being given by their eccentric anomalies α, β, γ, δ.

147

THE point P on an ellipse is the projection of Q on the circle on the transverse axis. P', Q' are the points diametrically opposite. Find the area of the parallelogram formed by the four points in terms of the eccentric anomaly of P.

148

FIND the area of the parallelogram formed by tangents at the same points.

149

FROM a point P whose distances from the foci of an ellipse are r, r' tangents are drawn to the curve. Find the angle between them.

150

SHOW that the equation to the pair of tangents from $x'y'$ to $b^2x^2 + a^2y^2 - a^2b^2$ may be written

$$a^2(x - x')^2 + b^2(y - y')^2 = (xy' - x'y)^2.$$

PUBLICATIONS RECEIVED.

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ERRATA.

Vol. III. p. 67, in last formula, for $-\frac{dh_2}{d\zeta}s - \theta h_3 u$, read $-\frac{dh_2}{d\zeta}u - \theta h_3 s$.

Vol. III. p. 72, for the factor $\frac{1-\nu}{\sqrt{(us)^3}}$ of one of the terms of each of the last three equations, read $\frac{1}{\sqrt{(us)^3}} - 1$.